

On the completeness of impulsive gravitational waves

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Abstract. We consider the impulsive limit of a class of space-times generalizing pp-waves that are given in the form $M = N \times \mathbb{R}_1^2$ where (N, h) is a Riemannian manifold of arbitrary dimension and M carries the line element $ds^2 = dh^2 + 2dudv + f(x)\delta(u)du^2$ with dh^2 the line element of N and δ the Dirac measure. We prove a completeness result for these space-times.

1. Introduction

Plane-fronted gravitational waves with parallel rays—pp-waves, for short—are defined by the existence of a covariantly constant null vector field \mathbf{k} and are usually associated with the line element in the so-called Brinkman form

$$ds^2 = 2dudv + dx_1^2 + dx_2^2 + H(x^1, x^2, u)du^2 \quad (1)$$

on \mathbb{R}^4 . These space-times model gravitational or electromagnetic waves and other forms of null matter and have been extensively studied (see e.g. [GP09, Ch. 17] and the literature cited therein). The geodesic null congruence with tangent \mathbf{k} is non-expanding, shear-free, and twist-free and the latter property implies the existence of a family of 2-surfaces perpendicular to \mathbf{k} which are interpreted as wave surfaces. Moreover, since $k^\mu_{;\mu}$ vanishes, they are planar and rays orthogonal to them are parallel.

It should be noted, however, that Brinkmann who studied these geometries in the context of conformal mappings of Einstein spaces ([Bri25]), also included a rotational term (rediscovered by Bonner [Bon70] and recently studied further under the name gyraton [Fro07]), as well as allowing for a general wave surface. Including the latter effect, i.e., allowing for a Riemannian manifold of arbitrary dimension as the wave surface we arrive at the following geometry (M, g) : Let (N, h) be a connected Riemannian manifold of dimension n , set $M = N \times \mathbb{R}_1^2$ and equip M with the line element

$$ds^2 = dh^2 + 2dudv + H(x, u)du^2. \quad (2)$$

Here dh^2 denotes the line element of (N, h) and u, v are global null-coordinates on the 2-dimensional Minkowski space \mathbb{R}_1^2 . Finally $H : M \times \mathbb{R} \rightarrow \mathbb{R}$ is a smooth function.

These models have been studied in a series of papers by J. Flores and M. Sanchez in part together with A. Candela ([CFS03, FS03, CFS04, FS06]) mainly focusing on causality and geodesics. These geometries allow to shed some light on some of the peculiar causal properties especially of plane waves (i.e., pp-waves (1) with $H(x^1, x^2, u) = h_{ij}(u)x^i x^j$), see e.g. [BEE96, Ch. 13]. They turn out to be caused by the high degree of symmetries of plane waves and the fact that the wave surfaces of (1) are flat \mathbb{R}^2 .

In [CFS03] space-times of the form (2) have been called (general) plane-fronted waves (PFW). However, by the geometric interpretation given above and by the analogy with pp-waves it seems more natural to us to call the space-times (2) *N-fronted waves with parallel rays (NPW)*, which we shall do from now on.

It turns out that the behaviour of H at spatial infinity, i.e., for „large x ” is decisive for many of the global properties of NPWs. In order to formulate precise statements we recall that one says that H behaves *subquadratically at spatial infinity* if there exist a fixed point $\bar{x} \in N$, continuous functions $0 \leq R_1, R_2$ and a continuous function $p < 2$ such that

$$H(x, u) \leq R_1(u)d^{p(u)}(x, \bar{x}) + R_2(u) \quad \forall (x, u) \in N \times \mathbb{R}. \quad (3)$$

Here d denotes the Riemannian distance function on N . Similarly we say that H behaves at most quadratically resp. superquadratically if $p \leq 2$ resp. $p > 2$. In [FS03] it has been shown that the causality of NPWs depends crucially on the exponent p in (3), with $p = 2$ being the critical case. In particular, NPWs are causal but not necessarily distinguishing, they are strongly causal if $-H$ behaves at most quadratically at spatial infinity and they are globally hyperbolic if $-H$ is subquadratic and N is complete. Similarly the global behaviour of geodesics in NPWs is governed by the behaviour of H at spatial infinity. From the explicit form of the geodesic equations it follows ([CFS03, Thm. 3.2]) that a NPW is complete iff N is complete and

$$D_{\dot{x}}^{(N)} \dot{x} = \frac{1}{2} \nabla_x H(x, s)$$

has complete trajectories. Here $D_{\dot{x}}^{(N)}$ is the induced covariant derivative on N and ∇_x denotes the spatial gradient. Applying classical results on complete vector fields (e.g. [AMR88, Thm. 3.7.15]) completeness of M follows for autonomous H (i.e., independent of u) in case H grows at most quadratic at spatial infinity. Clearly this implies completeness for at most quadratic sandwich waves, i.e., H compactly supported in u .

In this work we consider *impulsive* NPWs (INPWs), i.e., we set $H(x, u) = f(x)\delta(u)$ in (2), where $\delta(u)$ is the Dirac measure on the hypersurface $\{u = 0\}$. Impulsive *pp-waves* (for a summary see [GP09, Ch. 20]) have been introduced by Penrose using a „scissors-and-paste method” (e.g. [Pen72]) gluing two halves of Minkowski space along the null hypersurface $\{u = 0\}$ with a warp. On the other hand, impulsive pp-waves arise as ultrarelativistic limits of Kerr-Newman black holes, the prototype being the Aichelburg-Sexl geometry ([AS71]).

The distributional term in the metric of impulsive pp-waves and INPWs makes it a delicate matter to mathematically deal with these space-times; for a general account on distributional geometries in GR see [SV06]. Therefore impulsive pp-waves have been treated using the nonlinear distributional geometry ([GKOS01]) built upon algebras of generalized functions ([Col85]). In particular, geodesics in impulsive pp-wave space-times have been considered in [Bal97, Ste98], and in [KS99], where an existence and uniqueness result for the geodesic equations has been proved. From a global point of view these results imply that impulsive pp-waves are geodesically complete.

In this short note we prove a completeness result for INPWs with complete N . We do so without using any theory of nonlinear distributions leaving a detailed study of INPWs as distributional geometries to a subsequent paper. More precisely, we view INPWs as geometries with a small but finitely extended impulse: Let δ_ϵ be some smooth approximation of the Dirac-delta (i.e., $\delta_\epsilon \rightarrow \delta$ weakly as $\epsilon \rightarrow 0$) and for fixed $\epsilon > 0$ consider the metric

$$ds_\epsilon^2 = dh^2 + 2dudv + f(x)\delta_\epsilon(u)du^2 \quad (4)$$

on M , where f is an arbitrary smooth function on N . We will show that for any geodesic γ in M there is ϵ_0 small enough, such that γ can be defined for all values of an affine parameter provided $\epsilon \leq \epsilon_0$. Moreover the size of ϵ_0 for which the geodesic becomes complete can be explicitly estimated in terms of (derivatives of) f and the initial data of γ . Finally, we also show that the globally defined geodesics converge to the geodesics of the background $N \times \mathbb{R}_1^2$ which, however, have to be joined with a suitable warp at the shock hypersurface.

2. The geodesic equations for INPWs

In this section we derive the geodesic equations for INPWs and fix some notation to be used in the remainder of this work. We start by making precise the class of regularizations we use for the Dirac delta. We set $I := (0, 1]$.

Definition 2.1. *A net $(\delta_\epsilon)_{\epsilon \in I}$ of smooth functions on \mathbb{R} is called a strict delta net if it satisfies the following three properties.*

- (i) *The supports shrink to zero, $\text{supp}(\delta_\epsilon) \rightarrow \{0\}$ for $\epsilon \searrow 0$.*
- (ii) *The integrals converge to 1, $\int_{\mathbb{R}} \delta_\epsilon(x) dx \rightarrow 1$ for $\epsilon \searrow 0$.*
- (iii) *The L^1 -norms are uniformly bounded, $\exists K > 0 : \int_{\mathbb{R}} |\delta_\epsilon(x)| dx \leq K \quad \forall \epsilon \in I$.*

Observe that this is a very general class of approximations of δ . (Even although smoothness excludes "boxes", nets arbitrary close to "boxes" and even discontinuous regularizations are practically included by the fact that \mathcal{C}_c^∞ is dense in L^1 .) Without loss of generality we will always assume that $\text{supp}(\delta_\epsilon) \subseteq (-\epsilon, \epsilon)$ for all $\epsilon \in I$.

Now let $M = N \times \mathbb{R}_1^2$ be an INPW with N a connected n -dimensional and complete Riemannian manifold and let M be endowed with the line element (4), where $(\delta_\epsilon)_\epsilon$ is a strict delta net.

Denoting the Christoffel symbols of the Riemannian manifold (N, h) by $\Gamma^{(N)}$ one obtains the non-vanishing Christoffel symbols for M w.r.t. a coordinate system (x^1, \dots, x^n) of N and (u, v) null-coordinates of \mathbb{R}_1^2

$$\begin{aligned}\Gamma_{ij}^k &= \Gamma_{ij}^{(N)k} \quad \text{for all } 1 \leq i, j, k \leq n, \\ \Gamma_{uj}^v &= \Gamma_{ju}^v = \frac{1}{2} \frac{\partial f}{\partial x^j} \delta_\epsilon \quad \text{for all } 1 \leq j \leq n, \\ \Gamma_{uu}^v &= \frac{1}{2} f \dot{\delta}_\epsilon, \\ \Gamma_{uu}^k &= -\frac{1}{2} h^{km} \frac{\partial f}{\partial x^m} \delta_\epsilon.\end{aligned}$$

Since all Christoffel symbols of the form Γ_{jk}^u vanish we may use u as an affine parameter for the geodesics which hence obey the following set of $n+1$ equations

$$\ddot{v}_\epsilon = -\frac{\partial f(x_\epsilon)}{\partial x^j} \dot{x}_\epsilon^j \delta_\epsilon - \frac{1}{2} f(x_\epsilon) \dot{\delta}_\epsilon, \quad (5)$$

$$D_{\dot{x}_\epsilon}^{(N)} \dot{x}_\epsilon = \frac{1}{2} \nabla_x f(x_\epsilon) \delta_\epsilon. \quad (6)$$

Here $D^{(N)}$ and ∇_x denote the covariant derivative resp. the gradient w.r.t. h . First observe that equation (5) can be integrated once the second equation has been solved. So we have to concentrate on equation (6), which is just the perturbed geodesic equation on N with potential f and the non-autonomous term δ_ϵ . Moreover, since the latter vanishes for $|u| \geq \varepsilon$ the x -component of the geodesics on M will for large u coincide with the (unperturbed) geodesics on N . By completeness of N the question of completeness of M reduces to the question whether all perturbed geodesics on N that enter the regularization strip at $u = -\varepsilon$ also leave it at $u = \varepsilon$, that is whether the perturbed geodesics blow up before $u = \varepsilon$ or not.

Bearing this in mind we apply the following procedure to solve the geodesic equation on M as well as to address the problem of geodesic completeness of M . We fix $\epsilon > 0$ and impose initial data $x_0 \in N$, $\dot{x}_0 \in T_{x_0}N$ “long before” the shock at $u = -1$ and then follow the unperturbed Riemannian geodesic on N with this data, i.e., the solution of

$$D_{\dot{x}}^{(N)} \dot{x} = 0, \quad x(-1) = x_0, \quad \dot{x}(-1) = \dot{x}_0,$$

which we denote by $x[x_0, \dot{x}_0]$. By completeness of N this geodesic $x[x_0, \dot{x}_0]$ will reach the shock region at $u = -\varepsilon$ and until then it will also be a solution of the perturbed geodesic equation (6) with the same data, which we will denote by $x_\epsilon[x_0, \dot{x}_0]$. With this notation we have $x_\epsilon[x_0, \dot{x}_0] = x[x_0, \dot{x}_0]$ on $] -\infty, -\epsilon]$ and to continue $x_\epsilon[x_0, \dot{x}_0]$ into the shock region $|u| \leq \epsilon$ we consider the initial value problem

$$(6) \text{ with data } x_\epsilon(-\epsilon) = x[x_0, \dot{x}_0](-\epsilon), \quad \dot{x}_\epsilon(-\epsilon) = \dot{x}[x_0, \dot{x}_0](-\epsilon).$$

To prove that $x_\epsilon[x_0, \dot{x}_0]$ extends to all values of the parameter u we only have to show that the latter initial value problem possesses a solution denoted by $\tilde{x}_\epsilon[x_0, \dot{x}_0]$ until $u = \epsilon$, since for $u \geq \epsilon$ the right hand side of (6) vanishes and we are solving the (unperturbed) geodesic equation in the complete manifold N . That is, we only have to show that no blow-up occurs within the shock region $|u| \leq \epsilon$, which, in fact, will be done in the next section (at least for ϵ small enough). In total we will then have the global perturbed geodesic $x_\epsilon[x_0, \dot{x}_0]$

$$x_\epsilon[x_0, \dot{x}_0](u) = \begin{cases} x[x_0, \dot{x}_0] & u \leq -\epsilon \\ \tilde{x}_\epsilon[x_0, \dot{x}_0] & -\epsilon \leq u. \end{cases} \quad (7)$$

Finally, as observed above, once we have such a solution of the x -component of the geodesic in M the equation for v can be integrated to give a solution for all $u \in \mathbb{R}$ and we will use the following notation: for initial conditions $v_0, \dot{v}_0 \in \mathbb{R}$ we denote by $v[v_0, \dot{v}_0]$ the straight line $v[v_0, \dot{v}_0](u) = v_0 + \dot{v}_0(1 + u)$, i.e., a solution of (5) for $u \leq -\epsilon$ and similarly $v_\epsilon[v_0, \dot{v}_0]$ denotes a solution of (5) with $v_\epsilon[v_0, \dot{v}_0](-\epsilon) = v[v_0, \dot{v}_0](-\epsilon)$ and $\dot{v}_\epsilon[v_0, \dot{v}_0](-\epsilon) = \dot{v}_0$.

3. Completeness

We now show that for any geodesic in M we can choose ϵ sufficiently small such that the geodesic can be extended through the shock. More precisely we prove that (using the notation introduced above) the initial value problem

$$D_{\dot{x}_\epsilon}^{(N)} \dot{x}_\epsilon = \frac{1}{2} \nabla_x f(x_\epsilon) \delta_\epsilon, \quad x_\epsilon(-\epsilon) = x[x_0, \dot{x}_0](-\epsilon), \quad \dot{x}_\epsilon(-\epsilon) = \dot{x}[x_0, \dot{x}_0](-\epsilon) \quad (8)$$

has a local solution defined up to $u = \epsilon$ provided ϵ is small enough.

Proposition 3.1. *For all $x_0 \in N$, $\dot{x}_0 \in T_{x_0}N$ there exists ϵ_0 such that the initial value problem (8) has a solution $\tilde{x}_\epsilon[x_0, \dot{x}_0]$ defined up to $u = \epsilon$ provided $\epsilon \leq \epsilon_0$.*

The proof heavily rests on a fixed point argument which we provide in detail in Lemma A.2 in the Appendix. Here we only observe that this argument indeed provides the assertion of the Proposition.

Proof: We invoke Lemma A.2 with $b > 0$, $c > 0$, $F_1(y, z)^k := -\Gamma_{ij}^{k(N)}(y) z^i z^j$ (to express $D^{(N)}$ in local coordinates) and $F_2(y)^k := \frac{1}{2} h^{km}(y) \frac{\partial f}{\partial x^m}(y)$ which is just $\frac{1}{2} \nabla_x f$ in coordinates. Clearly F_1 and F_2 are smooth since f and h (and hence the Christoffel symbols) are assumed to be smooth. Hence Lemma A.2 guarantees existence of a solution $\tilde{x}_\epsilon[x_0, \dot{x}_0]$ of (8) until $u = \alpha - \epsilon$. So choosing $\epsilon_0 = \frac{\alpha}{2}$, the solution $\tilde{x}_\epsilon[x_0, \dot{x}_0]$ exists at least until $u = +\epsilon$, provided $\epsilon \leq \epsilon_0$. \square

Observe that Lemma A.2 also implies that the solution $\tilde{x}_\epsilon[x_0, \dot{x}_0]$ together with its first derivative $\dot{\tilde{x}}_\epsilon[x_0, \dot{x}_0]$ is uniformly bounded (in ϵ on $[-\epsilon, \epsilon]$). Moreover (A.2) gives an upper bound on ϵ_0 in terms of the initial velocity \dot{x}_0 and of the Christoffel symbols on N as well as of $\nabla_x f$ on a neighborhood of the data x_0 . Next we state our completeness result.

Theorem 3.2. *For all $x_0 \in N$, $\dot{x}_0 \in T_{x_0}N$ and all $v_0, \dot{v}_0 \in \mathbb{R}$ there exists ϵ_0 such that the solution $(x_\epsilon[x_0, \dot{x}_0], v_\epsilon[v_0, \dot{v}_0])$ of the geodesic equation (5,6) with initial data $x_\epsilon(-1) = x_0$, $\dot{x}_\epsilon(-1) = \dot{x}_0$, $v_\epsilon(-1) = v_0$, $\dot{v}_\epsilon(-1) = \dot{v}_0$ is defined for all $u \in \mathbb{R}$, provided $\epsilon \leq \epsilon_0$.*

Proof: Given x_0, \dot{x}_0 , Proposition 3.1 provides us with ϵ_0 such that the solution of (8) is defined for $u \in [-\epsilon, \epsilon]$ for all $\epsilon \leq \epsilon_0$. In this case we hence may define $x_\epsilon[x_0, \dot{x}_0]$ as in (7) for all $u \in \mathbb{R}$ so it only remains to integrate (5) twice to obtain a globally defined solution v_ϵ . Hence in total we obtain a unique globally defined geodesic $\mathbb{R} \ni u \mapsto (x_\epsilon[x_0, \dot{x}_0](u), v_\epsilon[v_0, \dot{v}_0](u))$. \square

We point out that α in the proof of Proposition 3.1 and hence ϵ_0 for which the geodesic is defined on all of \mathbb{R} depends on the choice of the initial data x_0 and \dot{x}_0 . Hence we can not, in general, obtain a global bound ϵ_0 such that for fixed $\epsilon \leq \epsilon_0$ the manifold M is geodesically complete. There are, however, two special cases where we actually obtain geodesic completeness of M for ϵ sufficiently small. First assume that N is compact. Then we obtain a globally defined ϵ_0 since x_0 varies in a compact set only and upon reparametrization we may achieve that $|\dot{x}_0| = 1$. On the other hand, if $-f$ behaves subquadratically (cf. (3)) then by the compactness of the support of δ_ϵ we may apply the results of [FS03] mentioned in the introduction to obtain completeness without even the need to invoke the fixed point argument.

However, one may say that “in the limit $\epsilon \rightarrow 0$ ” we obtain a geodesically complete manifold, hence one may say that IPNWs are geodesically complete irrespectively of the behaviour of the profile function f . This is in sharp contrast to the case of extended NPWs where completeness depends crucially on the behavior of H at “spatial infinity”: the role of the x -asymptotics of H becomes irrelevant in the impulsive limit.

However, the precise meaning of the completeness statement (i.e. the dependence of ϵ_0 on the data) is encoded in the formulation of our theorem above. A more straight forward completeness result for IPNWs can be provided using nonlinear distributional geometry ([GKOS01, KS02]) in the sense of J.F. Colombeau ([Col85]), and we will address this topic in a subsequent paper.

4. Limits

In this section we compute the limits of the global geodesics derived above as $\epsilon \rightarrow 0$. We start by analyzing the x -component and introduce some more notation in the same spirit as at the end of section 2. To denote the prospective limit of $x_\epsilon[x_0, \dot{x}_0]$ we have to paste together the solution $x[x_0, \dot{x}_0]$ of the unperturbed equation for $u < 0$ with an appropriate solution of the unperturbed equation for $u > 0$. To this end denote by $\tilde{x}[x_0, \dot{x}_0]$ the solution of the (unperturbed) geodesic equation on N with data $\tilde{x}(0) = x[x_0, \dot{x}_0](0)$ and $\dot{\tilde{x}}(0) = \dot{x}[x_0, \dot{x}_0](0) + \frac{1}{2}\nabla_x f(x[x_0, \dot{x}_0](0))$. Finally denote the prospective limit by

$$y[x_0, \dot{x}_0](u) := \begin{cases} x[x_0, \dot{x}_0](u) & u \leq 0 \\ \tilde{x}[x_0, \dot{x}_0](u) & u \geq 0. \end{cases} \quad (9)$$

Observe that $y[x_0, \dot{x}_0]$ is a continuous curve $\mathbb{R} \rightarrow N$ which is piece-wise smooth with a single break point at $u = 0$. Moreover it is not differentiable (in general) as we have

$$\begin{aligned} \lim_{u \nearrow 0} \dot{y}[x_0, \dot{x}_0](u) &= \dot{x}[x_0, \dot{x}_0](0), \\ \lim_{u \searrow 0} \dot{y}[x_0, \dot{x}_0](u) &= \dot{x}[x_0, \dot{x}_0](0) + \frac{1}{2} \nabla_x f(x[x_0, \dot{x}_0](0)). \end{aligned}$$

For simplicity we write $F_1(y, z)^k := -\Gamma_{ij}^{k(N)}(y) z^i z^j$ and $F_2^k = \frac{1}{2} \nabla_x^k f = \frac{1}{2} h^{kl} \frac{\partial f}{\partial x^l}$ as in the proof of Theorem 3.1 and start with an auxiliary result needed throughout the remainder of this section.

Lemma 4.1. *The global solution $x_\epsilon[x_0, \dot{x}_0]$ of (6) (defined in (7)) satisfies*

$$x_\epsilon[x_0, \dot{x}_0](\epsilon u) \rightarrow y[x_0, \dot{x}_0](0) = x[x_0, \dot{x}_0](0) \text{ uniformly on } [-1, 1] \text{ as } \epsilon \searrow 0.$$

Proof: To keep the notation transparent we abbreviate $x[x_0, \dot{x}_0]$ by x and $x_\epsilon[x_0, \dot{x}_0]$ by x_ϵ . We have

$$\sup_{u \in [-1, 1]} |x_\epsilon(\epsilon u) - x(0)| \leq \sup_{u \in [-1, 1]} \underbrace{|x_\epsilon(\epsilon u) - x(\epsilon u)|}_{=:(*)} + \sup_{u \in [-1, 1]} |x(\epsilon u) - x(0)|.$$

The second term goes to zero as $\epsilon \searrow 0$ since x is uniformly continuous on compact sets. To estimate the first term we integrate the differential equations for x_ϵ and x (see also the Appendix) to obtain

$$\begin{aligned} (*) &\leq \int_{-\epsilon}^{\epsilon u} \int_{-\epsilon}^s |F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) - F_1(x(r), \dot{x}(r))| dr ds + \int_{-\epsilon}^{\epsilon u} \int_{-\epsilon}^s |F_2(x_\epsilon(r))| |\delta_\epsilon(r)| dr ds \\ &\leq C\epsilon^2 + C\|\delta_\epsilon\|_{L^1} \epsilon \leq C\epsilon \rightarrow 0 \text{ } (\epsilon \searrow 0), \end{aligned}$$

where we have used that by Lemma A.2, x_ϵ and \dot{x}_ϵ are bounded independently of ϵ . \square

Proposition 4.2. *The global solution $x_\epsilon[x_0, \dot{x}_0]$ of (6) (defined in (7)) satisfies*

$$\begin{aligned} x_\epsilon[x_0, \dot{x}_0] &\rightarrow y[x_0, \dot{x}_0] \quad \text{uniformly on compact subsets of } \mathbb{R}, \\ \dot{x}_\epsilon[x_0, \dot{x}_0] &\rightarrow \dot{y}[x_0, \dot{x}_0] \quad \text{uniformly on compact subsets of } \mathbb{R} \setminus \{0\}. \end{aligned}$$

Proof: Again we write x for $x[x_0, \dot{x}_0]$ and x_ϵ for $x_\epsilon[x_0, \dot{x}_0]$ and similarly y for $y[x_0, \dot{x}_0]$ and \tilde{x}_ϵ for $\tilde{x}_\epsilon[x_0, \dot{x}_0]$. Without loss of generality we only consider the compact interval $[-1, 1]$. We distinguish three cases: $-1 \leq u \leq -\epsilon$, $-\epsilon \leq u \leq \epsilon$ and $\epsilon \leq u \leq 1$.

In the first case $x_\epsilon = x = y$ on $[-1, -\epsilon]$ (and hence $\dot{x}_\epsilon = \dot{x}$ on the same interval), since x_ϵ and x solve the same initial value problem. If $-\epsilon \leq u \leq \epsilon$ the result for x_ϵ follows immediately from Lemma 4.1 while for the derivative \dot{x}_ϵ there is nothing to prove in this case.

Finally, for $\epsilon \leq u \leq 1$ we observe that $x_\epsilon = \tilde{x}_\epsilon$ and $y = \tilde{x}$ solve the same differential equation but now with different initial conditions, namely $\tilde{x}_\epsilon(\epsilon)$, $\dot{\tilde{x}}_\epsilon(\epsilon)$, and $\tilde{x}(\epsilon)$ and $\dot{\tilde{x}}(\epsilon)$, respectively. By continuous dependence on the initial data we obtain

$$\max(|\tilde{x}_\epsilon(u) - \tilde{x}(u)|, |\dot{\tilde{x}}_\epsilon(u) - \dot{\tilde{x}}(u)|) \leq \max(|\tilde{x}_\epsilon(\epsilon) - \tilde{x}(\epsilon)|, |\dot{\tilde{x}}_\epsilon(\epsilon) - \dot{\tilde{x}}(\epsilon)|) e^L,$$

where L is a Lipschitz constant of F_1 on the compact image of $[0, 1]$ under \tilde{x} , $\dot{\tilde{x}}$, \tilde{x}_ϵ , $\dot{\tilde{x}}_\epsilon$ and it suffices to estimate the difference of the data. Indeed we have

$$|\tilde{x}_\epsilon(\epsilon) - \tilde{x}(\epsilon)| \leq |\tilde{x}_\epsilon(\epsilon) - \tilde{x}(0)| + |\tilde{x}(0) - \tilde{x}(\epsilon)| \rightarrow 0,$$

since the first term converges to zero by Lemma 4.1 and the second by continuity. Similarly we have

$$|\dot{\tilde{x}}_\epsilon(\epsilon) - \dot{\tilde{x}}(\epsilon)| \leq |\dot{\tilde{x}}_\epsilon(\epsilon) - \dot{\tilde{x}}(0)| + |\dot{\tilde{x}}(0) - \dot{\tilde{x}}(\epsilon)|,$$

where again the second term on the r.h.s. goes to zero by continuity. To estimate the first term we plug in the integral representation of $\dot{\tilde{x}}_\epsilon$ to obtain

$$\begin{aligned} |\dot{\tilde{x}}_\epsilon(\epsilon) - \dot{\tilde{x}}(0)| &\leq |\dot{\tilde{x}}_\epsilon(\epsilon) - \dot{x}(0) - F_2(x(0))| \\ &\leq |\dot{\tilde{x}}_\epsilon(-\epsilon) - \dot{x}(0)| + \int_{-\epsilon}^{\epsilon} |F_1(\tilde{x}_\epsilon(s), \dot{\tilde{x}}_\epsilon(s))| ds \\ &\quad + \left| \int_{-\epsilon}^{\epsilon} F_2(\tilde{x}_\epsilon(s)) \delta_\epsilon(s) ds - F_2(x(0)) \right|. \end{aligned}$$

Now the first term on the r.h.s. vanishes in the limit since $\dot{\tilde{x}}_\epsilon(-\epsilon) = \dot{x}(-\epsilon) \rightarrow \dot{x}(0)$. The second term goes to zero again by the uniform boundedness of \tilde{x}_ϵ and $\dot{\tilde{x}}_\epsilon$. To obtain the same conclusion for the third term we again take into account the uniform boundedness of \tilde{x}_ϵ and the fact that $(\delta_\epsilon)_\epsilon$ is a strict delta net. \square

Next we turn to the v -component and recall that $(u, v) \in \mathbb{R}_1^2$ and so we may work distributionally.

Proposition 4.3. *The global solution $v_\epsilon[v_0, \dot{v}_0]$ of (5) satisfies*

$$v_\epsilon[v_0, \dot{v}_0] \rightarrow v[v_0, \dot{v}_0] - \frac{1}{2} f(x(0))H - \left(\dot{x}^j(0) + \frac{1}{4} \nabla_x f^j(x(0)) \right) D_j f(x(0)) u_+,$$

where $u_+(u) = uH(u)$ denotes the kink function and we again have abbreviated $x[x_0, \dot{x}_0]$ by x .

Proof: In addition to the abbreviations x and x_ϵ used already above we write v for $v[v_0, \dot{v}_0]$ and v_ϵ for $v_\epsilon[v_0, \dot{v}_0]$. Since we have $v_\epsilon(u) = v_0 + \dot{v}_0(1 + u) + H * H * \ddot{v}_\epsilon(u)$ and since convolution is a separately continuous operation, it suffices to calculate the distributional limit of \ddot{v}_ϵ . Inserting the integral representation of \dot{x}_ϵ^j into equation (5) we obtain

$$\begin{aligned} \ddot{v}_\epsilon(u) = & \underbrace{-D_j f(x_\epsilon(u)) \delta_\epsilon(u) \dot{x}_\epsilon^j(-\epsilon)}_{(I)} - \frac{1}{2} \underbrace{D_j f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^u F_1^j(x_\epsilon(s), \dot{x}_\epsilon(s)) ds}_{(II)} \\ & - \frac{1}{2} \underbrace{D_j f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^u F_2^j(x_\epsilon(s)) \delta_\epsilon(s) ds}_{(III)} - \frac{1}{2} \underbrace{\frac{d}{du}(f(x_\epsilon(u)) \delta_\epsilon(u))}_{(IV)}. \end{aligned}$$

It is easily seen that $(I) \rightarrow \dot{x}^j(0)D_j f(x(0))\delta$ and that $(IV) \rightarrow f(x(0))\dot{\delta}$ in $\mathcal{D}'(\mathbb{R})$. On the other hand $(II) \rightarrow 0$ in $\mathcal{D}'(\mathbb{R})$ since for all test functions $\phi \in \mathcal{D}(\mathbb{R})$ we have (again using the uniform boundedness of x_ϵ and \dot{x}_ϵ)

$$\left| \int_{\mathbb{R}} \phi(u) D_j f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^u F_1^j(x_\epsilon(r), \dot{x}_\epsilon(r)) dr du \right| \leq 2\epsilon C \|\phi\|_\infty.$$

Finally, we show that $(III) \rightarrow \frac{1}{2}D_j f(x(0))F_2^j(x(0))\delta$. Indeed, we have

$$\begin{aligned} & \left| \int_{\mathbb{R}} \phi(u) D_j f(x_\epsilon(u)) \delta_\epsilon(u) \int_{-\epsilon}^u F_2^j(x_\epsilon(r)) \delta_\epsilon(r) dr du - \frac{1}{2} \phi(0) D_j f(x(0)) F_2^j(x(0)) \right| \\ & \leq \left| \int_{-\epsilon}^\epsilon \phi(u) \delta_\epsilon(u) \int_{-\epsilon}^u F_2^j(x_\epsilon(r)) \delta_\epsilon(r) dr (D_j f(x_\epsilon(u)) - D_j f(x(0))) du \right| \\ & \quad + \left| \int_{-\epsilon}^\epsilon \phi(u) \delta_\epsilon(u) \int_{-\epsilon}^u (F_2^j(x_\epsilon(r)) - F_2^j(x(0))) \delta_\epsilon(r) dr du \right| |D_j f(x(0))| \\ & \quad + \left| \int_{-\epsilon}^\epsilon \phi(u) \delta_\epsilon(u) \int_{-\epsilon}^u \delta_\epsilon(r) dr du - \frac{1}{2} \phi(0) \right| |D_j f(x(0))| |F_2^j(x(0))| \\ & \leq C \|\phi\|_\infty \sup_{u \in [-1,1]} |D_j f(x_\epsilon(\epsilon u)) - D_j f(x(0))| \\ & \quad + C \|\phi\|_\infty \sup_{u \in [-1,1]} |F_2^j(x_\epsilon(\epsilon u)) - F_2^j(x(0))| \\ & \quad + C \left| \int_{-\epsilon}^\epsilon \phi(u) \delta_\epsilon(u) \int_{-\epsilon}^u \delta_\epsilon(r) dr du - \frac{1}{2} \phi(0) \right|. \end{aligned}$$

Now the first and the second term converge to zero, again by Lemma 4.1. Finally, the integral term in the last line converges to zero by an elementary calculation. \square

Summing up we have shown that the x -component of the limit is continuous but has a kink at the shock hypersurface. The v -component, however, is not even continuous but has a jump at the shock in addition to a kink. The parameters of the kinks and the jump are given in terms of the profile function f and its derivatives at the point where the geodesic hits the shock hypersurface. So globally the geodesics on M are given by geodesics on the background $N \times \mathbb{R}_1^2$, which have to be joined suitably at the shock hypersurface.

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Appendix

In this appendix we detail the fixed point argument used in the proof of our main result. Our argument is built on a slightly sharper version of the Banach fixed point theorem (see [Wei52]). Indeed the integral operator A_ϵ used below to solve the initial

value problem is *not* a contraction on the naturally chosen Banach space X_ϵ and so the Banach fixed point theorem does not apply.

Theorem A.1. (*Weissinger's fixed point theorem*) *Let X be a nonempty closed subset of a Banach space $(E, \|\cdot\|)$. Moreover let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive real numbers $(a_n)_n$ and $A : X \rightarrow X$ a map with the property that*

$$\|A^n(u) - A^n(v)\| \leq a_n \|u - v\| \quad \forall u, v \in X \quad \forall n \in \mathbb{N}. \quad (\text{A.1})$$

Then A has a unique fixed point.

We may now state and proof our main technical result.

Lemma A.2. *Let $F_1 \in \mathcal{C}^\infty(\mathbb{R}^{2n}, \mathbb{R}^n)$, $F_2 \in \mathcal{C}^\infty(\mathbb{R}^n, \mathbb{R}^n)$, let $x_0, \dot{x}_0 \in \mathbb{R}^n$, let $b > 0$, $c > 0$ be given and let $(\delta_\epsilon)_\epsilon$ be a strict delta net with L^1 -bound $K > 0$. Define $I_1 := \{x \in \mathbb{R}^n : |x - x_0| \leq b\}$, $I_2 := \{x \in \mathbb{R}^n : |x - \dot{x}_0| \leq c + K\|F_2\|_{I_1, \infty}\}$ and $I_3 := I_1 \times I_2$. Moreover set*

$$\alpha := \min \left(1, \frac{b}{|\dot{x}_0| + \|F_1\|_{I_3, \infty} + K\|F_2\|_{I_1, \infty}}, \frac{c}{\|F_1\|_{I_3, \infty}} \right). \quad (\text{A.2})$$

Then the initial value problem

$$\begin{aligned} \ddot{x}_\epsilon &= F_1(x_\epsilon, \dot{x}_\epsilon) + F_2(x_\epsilon)\delta_\epsilon, \\ x_\epsilon(-\epsilon) &= x_0, \quad \dot{x}_\epsilon(-\epsilon) = \dot{x}_0, \end{aligned} \quad (\text{A.3})$$

has a unique solution x_ϵ on $J_\epsilon := [-\epsilon, \alpha - \epsilon]$ with $(x_\epsilon(J_\epsilon), \dot{x}_\epsilon(J_\epsilon)) \subseteq I_3$. In particular, both x_ϵ and \dot{x}_ϵ are bounded, uniformly in ϵ .

Proof: We consider the closed subset $X_\epsilon := \{x_\epsilon \in \mathcal{C}^\infty(J_\epsilon, \mathbb{R}^n) : x_\epsilon(J_\epsilon) \subseteq I_1, \dot{x}_\epsilon(J_\epsilon) \subseteq I_2\}$ of the Banach space $\mathcal{C}^1(J_\epsilon, \mathbb{R}^n)$ with norm $\|x\|_{\mathcal{C}^1} = \|x\|_{J_\epsilon, \infty} + \|\dot{x}\|_{J_\epsilon, \infty}$. We define the operator $A_\epsilon : X_\epsilon \rightarrow X_\epsilon$ by ($t \in J_\epsilon$)

$$A_\epsilon(x_\epsilon)(t) := x_0 + \dot{x}_0(t + \epsilon) + \int_{-\epsilon}^t \int_{-\epsilon}^s F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) dr ds + \int_{-\epsilon}^t \int_{-\epsilon}^s F_2(x_\epsilon(r)) \delta_\epsilon(r) dr ds.$$

First we show that the operator A_ϵ maps X_ϵ to itself. Let $x_\epsilon \in X_\epsilon$ and $t \in J_\epsilon$, then we have for the zero-order derivative of $A_\epsilon(x_\epsilon)$

$$\begin{aligned} |A_\epsilon(x_\epsilon)(t) - x_0| &\leq |\dot{x}_0|(t + \epsilon) + \int_{-\epsilon}^t \int_{-\epsilon}^s |F_1(x_\epsilon(r), \dot{x}_\epsilon(r))| dr ds + \int_{-\epsilon}^t \int_{-\epsilon}^s |F_2(x_\epsilon(r))| |\delta_\epsilon(r)| dr ds \\ &\leq \alpha |\dot{x}_0| + \alpha^2 \|F_1\|_{I_3, \infty} + \alpha \|F_2\|_{I_1, \infty} \|\delta_\epsilon\|_{L^1} \\ &\leq \alpha (|\dot{x}_0| + \|F_1\|_{I_3, \infty} + K\|F_2\|_{I_1, \infty}) \leq b, \end{aligned}$$

and for the first-order derivative

$$\begin{aligned} \left| \frac{d}{dt}(A_\epsilon(x_\epsilon))(t) - \dot{x}_0 \right| &\leq \int_{-\epsilon}^t |F_1(x_\epsilon(r), \dot{x}_\epsilon(r))| dr + \int_{-\epsilon}^t |F_2(x_\epsilon(r))| |\delta_\epsilon(r)| dr \\ &\leq \alpha \|F_1\|_{I_3, \infty} + \|F_2\|_{I_1, \infty} \|\delta_\epsilon\|_{L^1} \leq c + K\|F_2\|_{I_1, \infty}. \end{aligned}$$

At this point we claim that we can find a sequence of positive real numbers $(a_n)_{n \geq 2}$ such that $\sum_{n=2}^{\infty} a_n < \infty$ and

$$\|A_\epsilon^n(x_\epsilon) - A_\epsilon^n(y_\epsilon)\|_{C^1(J_\epsilon)} \leq a_n \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)}.$$

So let $\mathbb{N} \ni n \geq 2$, $x_\epsilon, y_\epsilon \in X_\epsilon$, and $t \in J_\epsilon$. Denoting by $[n \int_{-\epsilon}^t]$ the n -times iterated integral we obtain (with $\text{Lip}(F_i, I_j)$ a Lipschitz constant for F_i on I_j)

$$\begin{aligned} & |A_\epsilon^n(x_\epsilon) - A_\epsilon^n(y_\epsilon)| \\ & \leq [2n \int_{-\epsilon}^t] |F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) - F_1(y_\epsilon(r), \dot{y}_\epsilon(r))| d^{2n}r \\ & \quad + [2n \int_{-\epsilon}^t] |F_2(x_\epsilon(r)) - F_2(y_\epsilon(r))| |\delta_\epsilon(r)| d^{2n}r \\ & \leq \left(\text{Lip}(F_1, I_3) [2n \int_{-\epsilon}^t] d^{2n}r \right. \\ & \quad \left. + \text{Lip}(F_2, I_1) \|\delta_\epsilon\|_{L^1} [(2n-1) \int_{-\epsilon}^t] d^{2n-1}r \right) \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)} \\ & \leq \left(\text{Lip}(F_1, I_3) \frac{\alpha^{2n}}{(2n)!} + \text{Lip}(F_2, I_1) \|\delta_\epsilon\|_{L^1} \frac{\alpha^{2n-1}}{(2n-1)!} \right) \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)}. \end{aligned}$$

Furthermore for the derivative of A_ϵ^n we find that

$$\begin{aligned} & \left| \frac{d}{dt}(A_\epsilon^n(x_\epsilon))(t) - \frac{d}{dt}(A_\epsilon^n(y_\epsilon))(t) \right| \\ & \leq [(2n-1) \int_{-\epsilon}^t] |F_1(x_\epsilon(r), \dot{x}_\epsilon(r)) - F_1(y_\epsilon(r), \dot{y}_\epsilon(r))| d^{2n-1}r \\ & \quad + [(2n-1) \int_{-\epsilon}^t] |F_2(x_\epsilon(r)) - F_2(y_\epsilon(r))| |\delta_\epsilon(r)| d^{2n-1}r \\ & \leq \left(\text{Lip}(F_1, I_3) [(2n-1) \int_{-\epsilon}^t] d^{2n-1}r \right. \\ & \quad \left. + \text{Lip}(F_2, I_1) \|\delta_\epsilon\|_{L^1} [(2n-2) \int_{-\epsilon}^t] d^{2n-2}r \right) \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)} \\ & \leq \left(\text{Lip}(F_1, I_3) \frac{\alpha^{2n-1}}{(2n-1)!} + \text{Lip}(F_2, I_1) \|\delta_\epsilon\|_{L^1} \frac{\alpha^{2n-2}}{(2n-2)!} \right) \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)}. \end{aligned}$$

Summing up we obtain

$$\|A_\epsilon^n(x_\epsilon) - A_\epsilon^n(y_\epsilon)\|_{C^1(J_\epsilon)} \leq 4 \max \left(\text{Lip}(F_1, I_3), K \text{Lip}(F_2, I_1) \right) \frac{\alpha^{2n-2}}{(2n-2)!} \|x_\epsilon - y_\epsilon\|_{C^1(J_\epsilon)},$$

which proves our claim.

Now Weissinger's fixed point theorem provides us with the existence of a unique solution $x_\epsilon \in X_\epsilon$.

Finally, since $x_\epsilon, \dot{x}_\epsilon$ stay in I_1 respectively I_2 (which are defined independently of ϵ), x_ϵ and \dot{x}_ϵ are bounded by b and $c + K\|F_2\|_{I_1, \infty}$, respectively. \square

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